Resit Exam - Statistics 2019/2020

Date: January 31, 2020
Time: 18.45-21.45
Place: A. Jacobshal 02
Progress code: WISTAT-07

Rules to follow:

- This is a closed book exam. Consultation of books and notes is **not** permitted. You can use a simple (non-programmable) calculator.
- Do not forget to write your name and student number onto each paper sheet.
- There are 5 exercises and the number of points per exercise are indicated within boxes. You can reach 90 points.
- For derivations include the relevant equation(s) and/or a short description.
- **We wish you success with the completion of the exam!**

START OF EXAM

1. **Maximum Likelihood and Method of Moments estimator.** $\boxed{\textbf{15}}$
   Suppose that X is a discrete random variable with the sample space $S = \{0, 1, 2, 3\}$ and the following density (PDF), where $\theta \in [0, 1]$ is an unknown parameter.

$$p(x|\theta) = \begin{cases} 2\theta/3 & \text{for } x = 0 \\ \theta/3 & \text{for } x = 1 \\ 2(1-\theta)/3 & \text{for } x = 2 \\ (1-\theta)/3 & \text{for } x = 3 \end{cases}$$

   A sample of size $n = 10$ has been taken from such a distribution. The realisations are: $(x_1, \ldots, x_{10}) = (3, 0, 2, 1, 3, 2, 1, 0, 2, 1)$.

   (a) Derive the maximum likelihood (ML) estimator of θ.
   If applicable, check via the 2nd derivative whether it is really a maximum. $\boxed{5}$

   (b) Derive the methods of moments (MOM) estimator of θ. $\boxed{5}$

   (c) Check whether the ML estimator is unbiased or asymptotically unbiased. $\boxed{5}$

2. **Method of Moments.** $\boxed{\textbf{10}}$
   We have a sample $X_1, \ldots, X_n$ from a Binomial distribution with unknown parameters $k \in \mathbb{N}$ and $p \in [0, 1]$. Compute the Method of Moments estimators for k and p. The density (pdf) of the Binomial distribution is:

$$f_{k,p}(x) = \binom{k}{x} \cdot p^x \cdot (1-p)^{k-x} \qquad (x = 0, \ldots, k)$$

   The expectation is $E[X] = k \cdot p$ and the variance is $Var(X) = k \cdot p \cdot (1-p)$.

3. **Sample from Poisson distribution.** **20**

Let X_1, X_2, X_3 be a sample from a Poisson distribution with density (PDF):

$$p(x|\lambda) = e^{-\lambda} \cdot \frac{\lambda^x}{x!} \qquad (x \in \mathbb{N}_0)$$

where $\lambda > 0$ is an unknown parameter. The expectation is λ, the variance is λ, and $Y := X_1 + X_2 + X_3$ has a Poisson distribution with parameter 3λ.

(a) 2 Determine a sufficient statistic for λ.

(b) 3 Compute the Fisher information $I(\lambda)$ (for a sample of size $n = 1$).

(c) 2+3 Show that the estimator $\hat{\lambda}_\star = \frac{X_1+2X_2+3X_3}{6}$ is unbiased and check whether $\hat{\lambda}_\star$ attains the Cramer-Rao bound.

(d) 5 Use your result from (a) and the Rao-Blackwell theorem to get a new unbiased estimator $\hat{\lambda}_\diamond$ with $Var(\hat{\lambda}_\diamond) \leq Var(\hat{\lambda}_\star)$. Does $\hat{\lambda}_\diamond$ attain the Cramer-Rao bound?

(d) 5 Derive the uniform most powerful (UMP) test for $H_0 : \lambda = 1$ vs. $H_1 : \lambda = 3$ to the level $\alpha = 0.05$. (Let $q_{\lambda,\alpha}$ denote the α quantile of a Poisson distribution with parameter λ.)

4. **Sample from Rayleigh distribution.** **25**

Consider two independent identically Gaussian distributed random variables U and W with $U, W \sim \mathcal{N}(0, \sigma^2)$. $X := \sqrt{U^2 + W^2}$ has then a Rayleigh distribution with parameter $\sigma^2 > 0$. The density (PDF) of a Rayleigh distribution is:

$$p(x|\sigma^2) = \frac{x}{\sigma^2} \cdot \exp\{-\frac{x^2}{2\sigma^2}\} \qquad (x \geq 0)$$

Let $X_1, \ldots, X_n$ be a sample from a Rayleigh distribution.

(a) Derive the maximum likelihood (ML) estimator $\hat{\sigma}^2_{ML}$ of σ^2.
If applicable, check via the 2nd derivative whether it is really a maximum. 5

(b) Show that the ML estimator is unbiased. 5

(c) Show that the Fisher information (for sample size $n = 1$) is $I(\sigma^2) = \frac{1}{\sigma^4}$. 5

(d) Check whether the ML estimator attains the Cramer Rao bound. 5

(e) Assume $n = 9$ and $\hat{\sigma}^2_{ML} = 4$. Derive an asymptotic two-sided 0.95 confidence interval for σ^2. (Use the asymptotic efficiency of the ML estimator.) 5

HINT 1: Recall: For a sample $X_1, \ldots, X_n$ from a Gaussian $\mathcal{N}(0, 1)$ distribution, $Y = X_1^2 + \ldots + X_n^2$ has a χ^2_n distribution with $E[Y] = n$ and $Var(Y) = 2n$.
HINT 2: Recall two quantiles of the $\mathcal{N}(0, 1)$ distribution: $q_{0.025} \approx -2$, $q_{0.95} \approx 1.6$.

5. **Likelihood Ratio Test Statistic.** **20**

Let $X_1, \ldots, X_n$ be random sample from a distribution that depends on one parameter $\theta \in \mathbb{R}$, and assume that the support (= the sample space) does not depend on θ. The density (PDF) of the distribution $f(x|\theta)$ is twice continuously differentiable, and $\hat{\theta}_{ML}$ denotes the maximum likelihood (ML) estimator of θ. Let θ_0 be the true parameter.

(a) Give the second order Taylor expansion of the log-likelihood.

$$l(\theta) = \log\left(f(X_1, \ldots, X_N|\theta)\right)$$

around $\hat{\theta}_{ML}$ at the true value $l(\theta_0)$. 5

(b) Show that

$$-2\log\left(\frac{L(\theta_0)}{\max\limits_{\theta\in\Theta}\{L(\theta)\}}\right) \approx -(\hat{\theta}_{ML} - \theta_0)^2 \cdot l''(\hat{\theta}_{ML})$$

where $L(\theta) = f(X_1, \ldots, X_N|\theta)$ denotes the likelihood. 5

(c) Show that for $n \to \infty$:

$$\frac{1}{n} \cdot l''(\theta_0) \to -I(\theta_0)$$

where $l''(.)$ is the 2nd derivative of the log-likelihood, and $I(.)$ is the Fisher information for a sample of size $n = 1$. 10

END OF EXAM